TROPICAL RATIONAL EQUIVALENCE ON $\mathbb{R}^r$

LARS ALLERMANN AND JOHANNES RAU

ABSTRACT. We introduce an improved version of rational equivalence in tropical intersection theory which can be seen as a replacement of [AR07, chapter 8]. Using this new definition, rational equivalence is compatible with push-forwards of cycles. Moreover, we prove that every tropical cycle in $\mathbb{R}^r$ is equivalent to a uniquely determined affine cycle, called its *degree*.

This article can be seen as an additional chapter of our paper [AR07]. We stick to the definitions and notations introduced there.
As discussed in [AR07, remark 8.6] our previous definition of rational equivalence was not compatible with push-forwards of cycles. In this work we give a stronger definition of rational equivalence that is able to resolve this problem without losing other useful features. Moreover, using this new definition we are able to prove that the k-th Chow group $A_k(\mathbb{R}^r)$ of $\mathbb{R}^r$ is isomorphic to the group $Z_k^{\text{aff}}(\mathbb{R}^r)$ of affine k-cycles in $\mathbb{R}^r$.

Definition 1. Let C be a cycle and let D be a subcycle. We call D *rationally equivalent to zero on* C, denoted by $D \sim 0$, if there exists a morphism $f : C' \to C$ and a bounded rational function ϕ on C' such that

$$f_*(\phi \cdot C') = D.$$

Let D' be another subcycle of C. Then we call D and D' rationally equivalent if $D - D'$ is rationally equivalent to zero.

Lemma 2. *Let D be a cycle in C rationally equivalent to zero. Then the following holds:*

(a) *Let E be another cycle. Then $D \times E$ is also rationally equivalent to zero.*
(b) *Let φ be a rational function on C. Then $\varphi \cdot D$ is also rationally equivalent to zero.*
(c) *Let $g : C \to \widetilde{C}$ be a morphism. Then $g_*(D)$ is also rationally equivalent to zero.*
(d) *Assume $C = \mathbb{R}^r$ and let E be another cycle in $\mathbb{R}^r$. Then $D \cdot E$ is also rationally equivalent to zero (where "$\cdot$" denotes the intersection product of cycles in $\mathbb{R}^r$ introduced in* [AR07, definition 9.3]*).*
(e) *Assume that D is zero-dimensional. Then $\deg(D) = 0$.*

Proof. Let $f : C' \to C$ be a morphism and ϕ a bounded function on C' such that $f_*(\phi \cdot C') = D$. Then $f \times \mathrm{id} : C' \times E \to C \times E$ provides (a), restricting f to $f : f^*(\varphi) \cdot C' \to C$ provides (b) and composing f with g provides (c).
(d): The intersection product $D \cdot E$ is defined to be

$$\pi_*(\max\{x_1, y_1\} \cdots \max\{x_r, y_r\} \cdot (D \times E)),$$

where the x_i (resp. y_i) are the coordinates of the first (resp. second) factor of $\mathbb{R}^r \times \mathbb{R}^r$ and $\pi : \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}^r$ is the projection onto the first factor. Thus we can apply (a) – (c).
(e): In this case C' must be one-dimensional and we can apply [AR07, lemma 8.3], which shows that the degree of $\phi \cdot C'$ is zero. But pushing forward preserves degree. □

An easy example of rationally equivalent cycles are translations.

Lemma 3. *Let C be a cycle in $\mathbb{R}^r$ and let $C(v)$ denote the translation of C by an arbitrary vector $v \in \mathbb{R}^r$. Then the equation*

$$C(v) \sim C$$

holds.

Proof. Consider the cycle $C \times \mathbb{R}$ in $\mathbb{R}^r \times \mathbb{R}$ with morphism

$$\begin{aligned} f : \mathbb{R}^r \times \mathbb{R} &\to \mathbb{R}^r, \\ (x,t) &\mapsto x + t \cdot e_i, \end{aligned}$$

where e_i is the i-th unit vector in $\mathbb{R}^r$. For $\mu \in \mathbb{R}$ let ϕ_μ be the bounded function

$$\phi_\mu(x,t) = \begin{cases} 0 & t \leq 0 \\ t & 0 \leq t \leq \mu \\ \mu & t \geq \mu. \end{cases}$$

Then $f_*(\phi_\mu \cdot C \times \mathbb{R}) = C - C(\mu \cdot e_i)$, which proves the claim. $\square$

Definition 4. Let C be a cycle in $\mathbb{R}^r$ of codimension k. Then we define d_C to be the map

$$\begin{aligned} Z_k(\mathbb{R}^r) &\to \mathbb{Z}, \\ D &\mapsto \deg(C \cdot D). \end{aligned}$$

Lemma 5. *Let $C = [(X, \omega_X)]$ be a d-dimensional affine cycle in $\mathbb{R}^r$. Then there always exists a representative $(X', \omega_{X'})$ of C and a complete simplicial fan Θ such that $X' \subseteq \Theta$.*

Proof. Let $X_0 := X = \{\sigma_1, \ldots, \sigma_N\}$ and let $\sigma_i = \{x \in \mathbb{R}^r | f_1^\sigma(x) \geq 0, \ldots, f_{k_\sigma}^\sigma(x) \geq 0\}$. Moreover, let $Y_0 := \{\mathbb{R}^r\}$ and for $f \in \Lambda^\vee$ let

$$H_f := \{\{x | f(x) \geq 0\}, \{x | f(x) = 0\}, \{x | f(x) \leq 0\}\}.$$

For all $i = 1, \ldots, N$ we construct refinements

$$X_i := X_{i-1} \cap H_{f_1^\sigma} \cap \ldots \cap H_{f_{k_\sigma}^\sigma}$$

and

$$Y_i := Y_{i-1} \cap H_{f_1^\sigma} \cap \ldots \cap H_{f_{k_\sigma}^\sigma}$$

as described in [GKM07, 2.5(e)]. This construction yields fans X_N and Y_N with $X_N^{(k)} \subseteq Y_N^{(k)}$ for all k and $|X| = |X_N|$. Moreover, Y_N is a complete fan in $\mathbb{R}^r$. We can make Y_N into a simplicial fan by further subdividing its cones: Let $\Theta := Y_N$. If $\sigma \in \Theta^{(p)}$ is generated by vectors $v_1, \ldots, v_q$ then remove σ and add all cones $\mathbb{R}_{\geq 0} v_{i_1} + \ldots + \mathbb{R}_{\geq 0} v_{i_k}$ for $1 \leq k \leq p$ and $1 \leq i_1 < \ldots < i_k \leq q$ to Θ. Finally, we take $(X', \omega_{X'}) := (X, \omega_X) \cap \Theta$ as described in [GKM07, 2.11(b)]. $\square$

Lemma 6. *Let C_1 and C_2 be affine cycles in $\mathbb{R}^r$ with $C_1 \sim C_2$. Then $C_1 = C_2$.*

Proof. Note that $C_1 \sim C_2$ implies $d_{C_1} = d_{C_2}$ by lemma 2 (d) and (e). Hence it suffices to show the following: If $C = [(X, \omega_X)]$ is a tropical cycle with $d_C = 0$ then $C = 0$.
We prove this by induction on $d := \dim(X)$. For $d = 0$ the situation is trivial, as we have $X = d_X(\mathbb{R}^r) \cdot \{0\}$. Therefore $d_X = 0$ implies $C = [(X, \omega_X)] = 0$.
To prove the induction step, we first use lemma 5, which shows that we can assume that X is the d-skeleton of a complete simplicial fan Θ with certain (possibly zero) weights on the d-dimensional cones. We have to show that, if we assume $d_X = 0$, all these weights are actually zero. So let σ be a facet of X and let $v_1, \ldots, v_d$ denote primitive vectors that generate σ. As Θ is simplicial, for each $i = 1, \ldots d$ there exists a unique rational function φ_i on Θ which fulfills $\varphi_i(v_i) = 1$ and is identically zero on all other rays of Θ. We now want to compute the weight of $\tau := \langle v_1, \ldots, v_{d-1} \rangle_{\mathbb{R}_{\geq 0}}$ in the intersection product $\varphi_d \cdot X$. As a representative of the primitive vector $v_{\sigma/\tau}$ we can use $\frac{1}{|\Lambda_\sigma / \Lambda_\tau + \mathbb{Z} v_d|} v_d$ (it might not be an integer vector, but modulo V_τ, it is a primitive generator of σ). Now, as φ_d is identically zero on

all facets containing τ but σ (in particular, φ_d is identically zero on τ), the weight of τ can be computed to be

$$\omega_{\varphi_d \cdot X}(\tau) = \omega_X(\sigma) \frac{1}{|\Lambda_\sigma / \Lambda_\tau + \mathbb{Z} v_d|} \varphi_d(v_d) = \frac{\omega_X(\sigma)}{|\Lambda_\sigma / \Lambda_\tau + \mathbb{Z} v_d|}.$$

Our assumption $d_C = d_X = 0$ implies $d_{\varphi_d \cdot X} = 0$, as $(\varphi_d \cdot X) \cdot Y = X \cdot (\varphi_d \cdot Y)$ for all cycles Y of complementary dimension. We apply the induction hypothesis to $\varphi_d \cdot X$ and conclude that $\varphi_d \cdot X = 0$ and thus $\omega_{\varphi_d \cdot X}(\tau) = 0$. But our above computation shows that this implies $\omega_X(\sigma) = 0$ and therefore $C = [(X, \omega_X)] = 0$. □

Theorem 7. *Let C be a cycle in $\mathbb{R}^r$. Then there exists an affine cycle $\delta(X)$ in $\mathbb{R}^r$ with*

$$X \sim \delta(X).$$

Proof. Let (X_1, ω_{X_1}) be a representative of $C_1 := C$. Refining (X_1, ω_{X_1}) we may assume that every polyhedron $\sigma \in X_1$ is the convex hull of its 1-skeleton (see for example [Z95, 1.2 and 2.2]) and that every polyhedron $\sigma \in X_1$ contains at least one vertex $\sigma \supseteq P_\sigma \in X_1^{(0)}$.

The 1-skeleton of X_1 is a finite graph Γ with edges $X_1^{(1)} = \{\varepsilon_1, \ldots, \varepsilon_N\}$ and vertices $X_1^{(0)} = \{P_0, \ldots, P_M\}$. By lemma 3 we may assume that P_0 is the origin. On every edge ε_i of this graph we choose an orientation and a primitive direction vector $v_i \in \Lambda_{\varepsilon_i}$ respecting this orientation (see figure 1 (a)). Then for $i = 1, \ldots, N$ let $l_i \cdot ||v_i||$ be the length of the edge ε_i (we set $l_i = \infty$ if ε_i is unbounded).

Adjacency of the edges in the graph Γ yields a system of linear equations in the variables l_i having the entries of the vectors v_i as coefficients (see figure 1 (b)). As the system is solved by the given lengths $l_i \in \mathbb{R}_{>0}$ and all vectors v_i are integral there exists a positive and integral solution $l'_1, \ldots, l'_N$. Using these numbers l'_i we construct a polyhedral complex X_1^t, $t \in \mathbb{R}$ as follows: We keep the position of the point P_0 fixed and for all $i = 1, \ldots, N$ we change the length of the edge ε_i to $l_i + t \cdot l'_i$. For a given polyhedron $\sigma \in X_1$ this process yields a deformation σ^t of σ which is not necessarily a polyhedron, but that can be decomposed into polyhedra $\sigma_1^t, \ldots, \sigma_{p_t}^t$ (see figure 1 (c)). If such a polyhedron σ_j^t is of dimension $\dim(C)$, then we define its weight to be

$$\widetilde{\omega_{X_1^t}}(\sigma_j^t) := (-1)^{\delta(\sigma_j^t)} \cdot \omega_{X_1}(\sigma),$$

where $\delta(\sigma_j^t)$ is the number of values $t' \in \mathbb{R}$ between 0 and t such that at least one of the lengths $l_{i_q} + t' \cdot l'_{i_q}$ occurring in the boundary of σ_j^t is zero. We denote by $\widetilde{X_1^t}$ the set of all polyhedra σ_j^t for $\sigma \in X_1$ and by $\widetilde{\omega_{X_1^t}}$ the weight function on the polyhedra of maximal dimension. Refining and possibly merging some of the σ_j^t (we have to add up the weights of all merged polyhedra) yields a tropical polyhedral complex $(X_1^t, \omega_{X_1^t})$. Note that $(X_1^0, \omega_{X_1^0}) = (X_1, \omega_{X_1})$. Furthermore, for $\sigma \in X_1$ we can consider the set

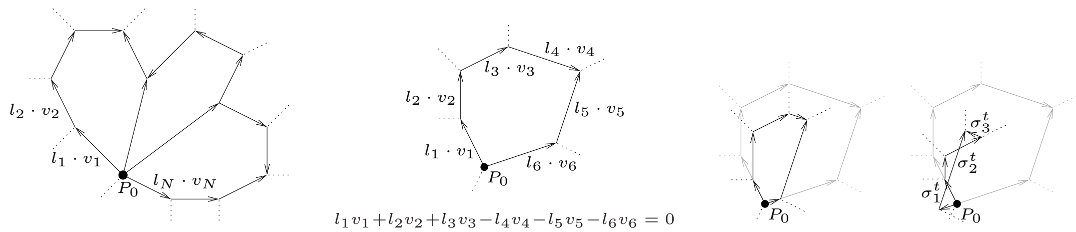
$$\widetilde{\sigma} := \bigcup_{t \in \mathbb{R}} \left(\bigcup_{j=1}^{p_t} \sigma_j^t \times \{t\} \right) \subseteq \mathbb{R}^r \times \mathbb{R}.$$

This set naturally splits up into polyhedra $\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_{s_\sigma}$. If a polyhedron $\widetilde{\sigma}_i$ is of maximal dimension we associate the weight $\widetilde{\omega_{X_1^t}}(\sigma_j^t)$ to it, where σ_j^t is a polyhedron containing a point in the relative interior of $\widetilde{\sigma}_i$ (this weight is obviously well-defined). We denote by $\widetilde{Z}$ the set $\{\widetilde{\sigma}_1, \ldots, \widetilde{\sigma}_{s_\sigma} | \sigma \in X_1\}$ and by $\widetilde{\omega_Z}$ the weight function on the polyhedra of maximal dimension. The choice of the weights $\widetilde{\omega_Z}(\widetilde{\sigma}_i)$ ensures that refining some of the $\widetilde{\sigma}_i$ yields a tropical polyhedral complex (Z, ω_Z).

Now, for $\mu \in \mathbb{R}$ let φ_μ be the rational function defined by

$$\varphi_\mu : \mathbb{R}^r \times \mathbb{R} \to \mathbb{R} : (x, t) \mapsto \max\{0, t\} - \max\{\mu, t\}.$$

Let $\widetilde{\sigma}_i \in Z$ be a polyhedron of maximal dimension and let (possibly after a refinement of X_1^t) $\sigma_j^t \subseteq \widetilde{\sigma}_i$ be a polyhedron of X_1^t of maximal dimension. As every polyhedron in X_1 contains at least one vertex,



(a) The oriented graph constructed from Γ.

(b) Adjacency in this graph yields linear equations.

(c) Two examples of the shrinking process.

FIGURE 1. Constructions described in the proof of theorem 7.

this property also holds for X_1^t and we can choose a vertex $P_{\sigma_j^t} \subseteq \sigma_j^t$. Let $P_{\sigma_j^{t+1}}$ be the translation of $P_{\sigma_j^t}$ in X_1^{t+1}. We have

$$P_{\sigma_j^{t+1}} - P_{\sigma_j^t} = \sum_{j=1}^{k} \pm l'_{i_j} v_{i_j}$$

for some $i_j \in \{1, \ldots, N\}$. Hence

$$\begin{pmatrix} \sum_{j=1}^{k} \pm l'_{i_j} v_{i_j} \\ 1 \end{pmatrix} \in \Lambda \times \mathbb{Z}$$

is a generator of $(\Lambda \times \mathbb{Z})_{\widetilde{\sigma_i}} / (\Lambda \times \mathbb{Z})_{\sigma^t}$ and we can deduce that

$$\varphi_\mu \cdot [(Z, \omega_Z)] = [(X_1^0, \omega_{X_1^0})] - [(X_1^\mu, \omega_{X_1^\mu})].$$

Now let $t_0 \in \mathbb{R}_{<0}$ be the largest value such that there exists an edge that has been shrunk to length 0, i.e. an edge $\varepsilon'_i \in (X_1^{t_0})^{(1)}$ with length $l_i + t_0 \cdot l'_i = 0$. We conclude that

$$\varphi_{t_0} \cdot [(Z, \omega_Z)] = C_1 - C_2,$$

where $C_2 := [(X_1^{t_0}, \omega_{X_1^{t_0}})]$ can be seen as the cycle $C = C_1$ with at least one bounded polyhedron shrunk to one dimension less.

We repeat the whole process until all bounded polyhedra are shrunk to a point, i.e. until we obtain an affine cycle C_p. By construction we have

$$C = C_1 \sim C_2 \sim \ldots \sim C_p,$$

which proves the claim. □

Definition 8. Let C be a cycle in $\mathbb{R}^r$. We define the *recession cycle or degree of* C, denoted by $\delta(C)$, to be the affine cycle equivalent to C. This affine cycle exists by theorem 7 and is unique by lemma 6.

Remark 9. Let σ be a polyhedron in $\mathbb{R}^r$. We define the *recession cone of* σ to be

$$\operatorname{rc}(\sigma) := \{v \in \mathbb{R}^r | x + \mathbb{R}_{\geq 0} v \subseteq \sigma \forall x \in \sigma\} = \{v \in \mathbb{R}^r | \exists x \in \sigma \text{ s.t. } x + \mathbb{R}_{\geq 0} v \subseteq \sigma\}.$$

The two sets coincide as σ is closed and convex.

Let C be a d-dimensional cycle in $\mathbb{R}^r$ with representative (X, ω_X) and let

$$\widetilde{R(C)} := \{\operatorname{rc}(\sigma) | \sigma \in X\}.$$

By removing all cones of $\widetilde{R(C)}$ that are not contained in a d-dimensional cone and by subdividing the remaining cones we can make this set into a fan $R(C)$ of pure dimension d. To every cone $\sigma \in R(C)^{(d)}$ we associate the weight

$$\omega_{R(C)}(\sigma) := \sum_{\substack{\sigma' \in X \\ \sigma \subseteq \operatorname{rc}(\sigma')}} \omega_X(\sigma').$$

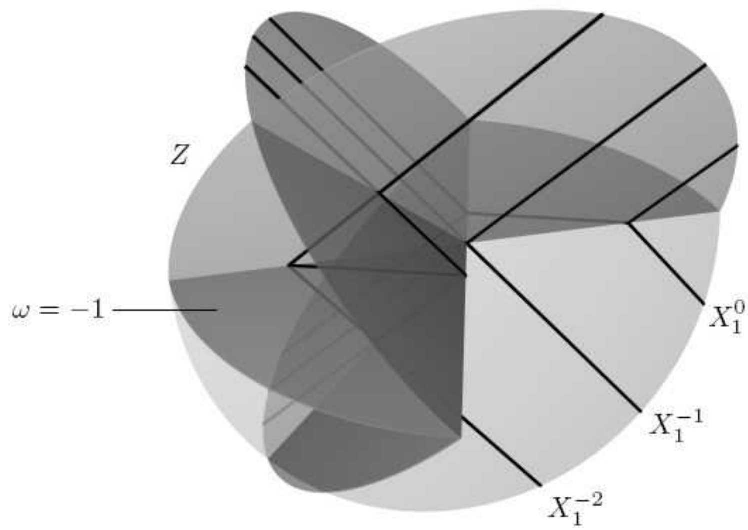


FIGURE 2. An example of a cycle Z as constructed in theorem 7.

The proof of theorem 7 indeed shows that

$$\delta(C) = [(R(C), \omega_{R(C)})]$$

holds.

Theorem 10. *Let C, D be two tropical cycles in $\mathbb{R}^r$. Then the following are equivalent:*

i) $C \sim D$
ii) $d_C = d_D$
iii) $\delta(C) = \delta(D)$

Proof. i) $\Rightarrow$ ii) follows from lemma 2 (d) and (e). iii) $\Rightarrow$ i) is an immediate consequence of theorem 7. ii) $\Rightarrow$ iii) follows from theorem 7, i) $\Rightarrow$ ii) and lemma 6. □

Remark 11. In other words, the above theorem says: Rational equivalence, numerical equivalence and "having the same degree" coincide.

Theorem 12 (General Bézout's theorem)**.** *Let C, D be two tropical cycles in $\mathbb{R}^r$. Then*

$$\delta(C \cdot D) = \delta(C) \cdot \delta(D).$$

Proof. We apply theorem 7 and get

$$\delta(C \cdot D) \sim C \cdot D \sim \delta(C) \cdot \delta(D)$$

(the second equivalence also uses lemma 2 (d)). By lemma 6 two rationally equivalent affine cycles are equal. □

REFERENCES

[AR07] Lars Allermann, Johannes Rau, *First steps in tropical intersection theory*, preprint arxiv:0709.3705.

[GKM07] Andreas Gathmann, Michael Kerber, Hannah Markwig, *Tropical fans and the moduli spaces of tropical curves*, preprint arxiv:0708.2268.

[Z95] Günter M. Ziegler, *Lectures on Polytopes*, Springer, 1995.

LARS ALLERMANN, FACHBEREICH MATHEMATIK, TECHNISCHE UNIVERSITÄT KAISERSLAUTERN, POSTFACH 3049, 67653 KAISERSLAUTERN, GERMANY

E-mail address: allerman@mathematik.uni-kl.de

JOHANNES RAU, FACHBEREICH MATHEMATIK, TECHNISCHE UNIVERSITÄT KAISERSLAUTERN, POSTFACH 3049, 67653 KAISERSLAUTERN, GERMANY

E-mail address: jrau@mathematik.uni-kl.de